Orbital Motion in the Theory of General Relativity

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According to the general theory of relativity, a masspoint describes a geodesic curve in four-dimensional spacetime. In particular, a description of the geodesic for Schwarzschild's line element yields the familiar equations of motion which result in the well-known advance of the perihelion of Mercury. These equations are separated here into the Newtonian equations of motion plus additional terms, which are interpreted as perturbing forces on the Newtonian motion and are resolved into radial and tangential perturbative accelerations. Variations in the classical Keplerian orbital elements are developed, and the secular or nonperiodic motions are derived by averaging the variations with respect to the mean anomaly. Besides the usual expression for the advance of the perihelion, a secular perturbation in the mean longitude is obtained with magnitude proportional to the square of the eccentricity. Short-period variations in the elements are given also. These are expressed as Fourier series in the mean anomaly with the coefficients carried to the second order in the eccentricity.

Introduction

In recent years there have been considerable advances in obtaining radio-tracking data accuracies, even to the point where mention of velocity measurements good to 0.001 m/sec at considerable distances is not uncommon. In anticipation of these and even greater accuracies it would seem that a complete perturbational analysis of the general theory of relativity is in order. Hopefully, then, this theory could be compared with radio observations of natural and artificial celestial objects.

It should be noted that there is considerable controversy not only over the interpretation of the coordinates in the Schwarzschild line element but also in the solution of the field equations of general relativity, e.g., see Fock.[1] In this article, then, the purpose is not to present a description of the motion that will solve dogmatically the problem for all time, but instead to provide a technique for quickly deriving the secular and short-period relativistic variations in the orbital elements when a particular line element is subjected to comparison with observation. Hopefully, the forthcoming radio observations of the planets, particularly Mercury as well as the continually improved data obtained from artificial objects, will prove useful at least in discovering obviously invalid line elements.

It seems necessary to emphasize this point because of remarks made by W. A. Mersman at the November 1962 Annual Meeting of the American Rocket Society in Los Angeles. In addition, note that the perturbative accelerations can be obtained from any line element of interest, and the variations in any directly or indirectly measured quantity follow by formulas such as those given by Herrick;[2] or, for example, the accelerations can be projected on a Cartesian system of axes, and in this way the relativity effects are included in, say, a Cowell integration of the equations of motion.

With respect to the extensive work in the past on relating general relativity to classical celestial mechanics, the contributions of de Sitter[3] and Chazy[4] certainly should be mentioned. However, because of the availability of optical data only, the secular terms in the elements have been emphasized with corresponding discussions of the divergences between the classical theory and optical observations of the planets and the moon. Here, again, in deriving the secular terms the Krylov-Bogoliubov method of averaging was unavailable in the older works, and the illustration of this technique presented here should be noted. Also it would seem that with radio observations the periodic terms, which are unavailable in the literature, might become important.

Finally, some mention has been made of closed-form solutions to the relativistic equations of motion. These solutions, even if available for every considered line element, do not offer any particular advantage in the problems considered here because a comparison with the data would eventually require an expansion of the solution in a form suitable for computation.

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Perturbative Equations

In this article variations in the two-body Keplerian orbital elements are developed. The motion is planar so that the variations di/dt and $d\Omega/dt$ in the inclination i and longitude of the ascending node Ω are zero. However, the other four elements all are affected. The time variations of the semimajor axis a, eccentricity e, argument of the perifocus ω, and mean anomaly phase angle χ(σ in Moulton's notation) are given in terms of R and S, the respective radial and tangential perturbative accelerations.[5]

$$\frac{da}{dt} = \frac{2a^2}{(\mu p)^{1/2}}\left(e \sin vR + \frac{p}{r} S\right) \tag{1}$$

$$\frac{de}{dt} = \frac{r}{(\mu p)^{1/2}}\left\{\frac{p}{r} \sin vR + \left[\left(1 + \frac{p}{r}\right)\cos v + e\right]\right\} S \tag{2}$$

$$\frac{d\omega}{dt} = -\frac{p^{1/2}}{e\mu^{1/2}}\left[\cos vR - \left(1 + \frac{r}{p}\right)\sin vS\right] \tag{3}$$

$$\frac{d\chi}{dt} = \frac{r}{e\mu^{1/2}}\left[\left(\frac{p}{r}\cos v - 2e\right)R - \left(1 + \frac{p}{r}\right)\sin vS\right] \tag{4}$$

The angle v is the true anomaly, μ is the gravitational constant, and p is the semilatus rectum.

$$p = a(1 - e^2) \tag{5}$$

The radius r is given by

$$r = p/(1 + e \cos v) \tag{6}$$

To obtain the perturbative accelerations R and S, the Einstein equations of motion are written in the polar form of Schwarzschild:[6]

$$\frac{d^2r}{ds^2} - \frac{1}{2\gamma}\frac{d\gamma}{dr}\left(\frac{ds}{dr}\right)^2 - r\gamma\left(\frac{dv}{ds}\right)^2 + \frac{1}{2}c^2\gamma\frac{d\gamma}{dr}\left(\frac{dt}{ds}\right)^2 = 0 \tag{7}$$

$$\frac{d}{ds}\left(r^2\frac{dv}{ds}\right) = 0 \tag{8}$$

$$\frac{d^2t}{ds^2} + \frac{1}{\gamma}\frac{d\gamma}{dr}\frac{dt}{ds}\frac{dr}{ds} = 0 \tag{9}$$

where

$$\gamma = 1 - (2\mu/c^2r) \tag{10}$$

The just mentioned equations are interpreted such that the coordinates r, v, and t correspond to the polar coordinates and time of the Newtonian theory. There is no assurance that this is so. For example, it is not clear that the coordinate t in Schwarzschild's line element can be equated with the independent time variable used in the description of planetary motions.‡ At present, it can be claimed only that, with respect to these motions, a perturbational analysis of Eqs. (7–9) leads to a successful description of the non-Newtonian advance of the perihelion of Mercury, at least within the accuracy of the observations. Beyond this measurement the validity of the assumptions made here can be tested only by comparing the results of the complete perturbational analysis with future optical and radio observations. This entire problem of the interpretation of the line element is discussed by Eddington[7] and more recently by McVittie.[8]

The first integral of Eq. (9) is

$$dt/ds = k/\gamma \tag{11}$$

with k the constant of integration. By means of Eq. (11), Eqs. (7) and (8) can be written in terms of time derivatives only. Equation (8) is simply

$$\frac{1}{r}\frac{d}{dt}\left(r^2\frac{dv}{dt}\right) = \frac{r}{\gamma}\frac{dv}{dt}\frac{d\gamma}{dt} \tag{12}$$

Thus, the Newtonian angular momentum is no longer a constant of the motion, although the angular momentum divided by γ is constant.

The term on the left of Eq. (12) is the tangential acceleration on the particle subjected to the differential equations (7–9). In Newtonian motion this acceleration is zero, so for the motion considered here the term on the right of Eq. (12) is the tangential perturbative acceleration S.

$$S = (r/\gamma)(dv/dt)(d\gamma/dt) \tag{13}$$

where from Eq. (10)

$$d\gamma/dt = (2\mu/c^2r^2)(dr/dt) \tag{14}$$

From this point on, the perturbations are developed only through terms in $1/c^2$. Because $d\gamma/dt$ is already of order $1/c^2$, it is necessary to include only zero-order terms in $1/\gamma$ and rdv/dt to obtain the perturbative acceleration S through $1/c^2$ terms. Thus,

$$\gamma = 1 + 0(1/c^2) \tag{15}$$

and from the two-body solution

$$r(dv/dt) = (1/r)(\mu p)^{1/2} + 0(1/c^2) \tag{16}$$

so that

$$S = (2\mu/c^2)[(\mu p)^{1/2}/r^3](dr/dt) \tag{17}$$

‡ The authors wish to express their gratitude to H. Lass for pointing out the various interpretations of general relativity.

while to all orders of $1/c^2$, the radial rate is given by

$$dr/dt = (\mu^{1/2}/p^{1/2})e\sin v \tag{18}$$

and therefore

$$S = (2\mu^2/c^2)(e\sin v/r^3) \tag{19}$$

The radial-perturbative acceleration can be obtained from Eq. (7) by recognizing that R plus the two-body Newtonian acceleration must equal the total radial acceleration. Thus,

$$(d^2r/dt^2) - r(dv/dt)^2 = -(\mu/r^2) + R \tag{20}$$

As with Eq. (8), Eq. (7) now is expressed in terms of time derivatives

$$\frac{d^2r}{dt^2} - \gamma r\left(\frac{dv}{dt}\right)^2 = -\frac{\gamma\mu}{r^2} + \frac{3\mu}{\gamma c^2r^2}\left(\frac{dr}{dt}\right)^2 \tag{21}$$

and, again, to order $1/c^2$

$$R = (\mu^2e/c^2r^2p)(3e - 2\cos v - 5e\cos^2 v) \tag{22}$$

Now, substituting formulas (19) and (22) in Eqs. (1–4),

$$\frac{da}{dt} = \frac{4\mu^2a^2}{c^2p(\mu p)^{1/2}}\frac{e\sin v}{r^2}\left(\frac{p}{r} + \frac{3}{2}e^2\sin^2 v\right) \tag{23}$$

$$\frac{de}{dt} = \frac{\mu^2}{c^2(\mu p)^{1/2}}\frac{e\sin v}{r^2}(5e + 2\cos v - 3e\cos^2 v) \tag{24}$$

$$\frac{d\omega}{dt} = \frac{\mu^2}{c^2(\mu p)^{1/2}}\frac{1}{r^2}(4 - e\cos v - 2\cos^2 v + 3e\cos^3 v) \tag{25}$$

$$\frac{d\chi}{dt} = \frac{\mu^2}{c^2(\mu a)^{1/2}}\frac{1}{r^2}\left[\left(\cos v - 2e\frac{r}{p}\right)(3e - 2\cos v - 5e\cos^2 v) - 2\left(1 + \frac{p}{r}\right)\sin^2 v\right] \tag{26}$$

Secular Motions

The secular motion in the four elements can be obtained by averaging the variations in formulas (23–26) with respect to the mean anomaly. Thus, for example, the average of da/dt is

$$\overline{\frac{da}{dt}} = \frac{1}{2\pi}\int_0^{2\pi}\frac{da}{dt}dM \tag{27}$$

or, in terms of the true anomaly,

$$\overline{\frac{da}{dt}} = \frac{1}{2\pi a^2(1 - e^2)^{1/2}}\int_0^{2\pi} r^2\frac{da}{dt}dv + 0\left(\frac{1}{c^4}\right) \tag{28}$$

Therefore, the average of the four variations is

$$\overline{da}/dt = 0 \tag{29}$$

$$\overline{de}/dt = 0 \tag{30}$$

$$\overline{d\omega}/dt = 3\mu n/c^2p \tag{31}$$

$$\overline{d\chi}/dt = -[3\mu n/ac^2(1 - e^2)^{1/2}][3 - 2(1 - e^2)^{1/2}] \tag{32}$$

The mean angular motion n is given by

$$n = \mu^{1/2}/a^{3/2} \tag{33}$$

The expression (31) for the motion of the perifocus is in agreement with the usual description and accounts for the discrepancy in the motion of the perihelion of Mercury from the Newtonian theory. The secular motion in χ is found to be opposite to that in ω, and for a circular orbit ($e = 0$) the motions exactly cancel. This implies that for circular orbits there is no detectable secular perturbation; furthermore, because the perturbation in the mean longitude L is the sum of the perturbations in ω and χ, the secular motion of L is

$$\frac{\overline{dL}}{dt} = n + \frac{\overline{d\omega}}{dt} + \frac{\overline{d\chi}}{dt} = n + e^2\frac{\overline{d\omega}}{dt}\left[\frac{1 - 2(1 - e^2)^{1/2}}{1 + (1 - e^2)^{1/2}}\right] \quad (34)$$

or, through the second order in the eccentricity,

$$\overline{dL}/dt = n - \tfrac{1}{2}e^2(\overline{d\omega}/dt) \quad (35)$$

This result should be compared with that of Bogorodskii,[9] who obtains a perturbation in χ which is three times that of Eq. (32) for circular motion. Such an increase in the mean motion would seem unrealistic. For the planet Mercury, Eqs. (31) and (35) yield $e = 0.206$, $\overline{d\omega}/dt = 42.5$ (sec of arc/century), and $(\overline{dL}/dt) - n = -0.9$ (sec of arc/century).

Periodic Motions

Approximate periodic motions now are obtained in the four considered elements. First Eqs. (23–26) are put in the form of harmonic series in the true anomaly.

$$\frac{da}{dt} = \frac{4\mu^2a^2e}{c^2p(\mu p)^{1/2}}\left(\frac{p}{r^3}\sin v + \frac{9}{8}\frac{e^2\sin v}{r^2} - \frac{3}{8}\frac{e^2\sin 3v}{r^2}\right) \quad (36)$$

$$\frac{de}{dt} = \frac{\mu^2 e}{c - (\mu p)^{1/2}}\left(\frac{17}{4}\frac{e\sin v}{r^2} + \frac{\sin 2v}{r^2} - \frac{3}{4}\frac{e\sin 3v}{r^2}\right) \quad (37)$$

$$\frac{d\omega}{dt} = \frac{\mu^2}{c^2(\mu p)^{1/2}}\left(\frac{3}{r^2} + \frac{5}{4}\frac{e\cos v}{r^2} - \frac{\cos 2v}{r^2} + \frac{3}{4}\frac{e\cos 3v}{r^2}\right) \quad (38)$$

$$\frac{d\chi}{dt} = \frac{\mu^2}{c^2(\mu a)^{1/2}}\left(-\frac{e^2}{pr} - \frac{2}{r^2} - \frac{p}{r^3} + \frac{4e\cos v}{pr} - \frac{3}{4}\frac{e\cos v}{r^2} + \frac{5e^2\cos 2v}{pr} + \frac{p\cos 2v}{r^3} - \frac{5}{4}\frac{e\cos 3v}{r^2}\right) \quad (39)$$

Using the tables of Cayley,[10] Eqs. (36–39) are expanded as Fourier series in the mean anomaly. Through the second order in the eccentricity the periodic terms are

$$\frac{da}{dM} = \frac{4e\mu}{c^2}\left(\sin M + \frac{5}{2}e\sin 2M\right) \quad (40)$$

$$\frac{de}{dM} = \frac{\mu e}{c^2 a}\left(\frac{13}{4}e\sin M + \sin 2M + \frac{9}{4}e\sin 3M\right) \quad (41)$$

$$\frac{d\omega}{dM} = \frac{\mu}{c^2a}\left[\frac{33}{4}e\cos M - \left(1 - \frac{23}{2}e^2\right)\cos 2M - \frac{9}{4}e\cos 3M - \frac{7}{2}e^2\cos 4M\right] \quad (42)$$

$$\frac{d\chi}{dM} = -\frac{\mu}{c^2a}\left(\frac{17}{4}e\cos M - (1 - e^2)\cos 2M - \frac{9}{4}e\cos 3M - \frac{7}{2}e^2\cos 4M\right) \quad (43)$$

The short-period variations are therefore

$$\delta a = -(4e\mu/c^2)[\cos M + \tfrac{5}{4}e\cos 2M] \quad (44)$$

$$\delta e = -(\mu e/2ac^2)[\tfrac{13}{2}e\cos M + \cos 2M + \tfrac{3}{2}e\cos 3M] \quad (45)$$

$$\delta\omega = \mu/2ac^2[\tfrac{33}{2}e\sin M - (1 - \tfrac{33}{2}e^2)\sin 2M - \tfrac{3}{2}e\sin 3M - \tfrac{7}{4}e^2\sin 4M] \quad (46)$$

$$\delta\chi = -(\mu/2ac^2)[\tfrac{17}{2}e\sin M - (1 - e^2)\sin 2M - \tfrac{3}{2}e\sin 3M - \tfrac{7}{4}e^2\sin 4M\omega] \quad (47)$$

The short-period perturbative variation in the mean longitude is

$$\delta L = \int(\delta n/n)dM + \delta\omega + \delta\chi \quad (48)$$

or

$$\delta L = (\mu e/ac^2)(10\sin M + 9e\sin 2M) \quad (49)$$

and again there is no detectable perturbation for circular motion.

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